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A Note on the Approximation of Functions of Several Variables by Sums of Functions of One Variable*

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For a class of functions of several variables, which includes the continuous functions, we show that there exists a sum of functions of one variable that minimizes the distance from the given function to the space of such sums. For functions of two variables we show that such a minimizing sum may be constructed by an iterative scheme.

I. INTRODUCTION

Let $\{\Omega_i\}_{i=1}^m$ be compact Hausdorff topological spaces endowed with positive regular Borel measures $\{\mu_i\}_{i=1}^m$. Let $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_m$ and $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_m$. Let $L_\infty(\Omega)$ denote the Banach space of μ-essentially bounded real-valued functions on Ω with the essential supremum norm. Let $S(\Omega)$ denote the subspace of $L_\infty(\Omega)$ consisting of sums of the form $\sum_{i=1}^m \phi_i$, with $\phi_i \in L_\infty(\Omega_i)$.

Let $S(\Omega)$ denote the subspace of $L^\infty(\Omega)$ given by

$$S(\Omega) = \left\{ f \in L^\infty(\Omega) \,|\, f(x) = \sum_{k=1}^m \phi_k(x_k), \phi_k \in L^\infty(\Omega_k) \right\}. \tag{1.1}$$

We show in the Appendix that $S(\Omega)$ is a closed subspace of $L^\infty(\Omega)$.

We let $K(\Omega)$ denote the closure in $L_\infty(\Omega)$ of the set of all finite sums of the form $\sum_{k=1}^M \prod_{j=1}^m \phi_{kj}$, where $\phi_{kj} \in L_\infty(\Omega_j)$ for $1 \leqslant j \leqslant m$ and $1 \leqslant k \leqslant M$. As products and sums of functions in $L_\infty(\Omega_j)$ are also in $L_\infty(\Omega_j)$, K is a closed subalgebra of $L_\infty(\Omega)$. In fact, $K(\Omega)$ is the smallest closed subalgebra of $L_\infty(\Omega)$ containing $S(\Omega)$. We note that $C(\Omega) \subset K(\Omega)$; here $C(\Omega)$ denotes

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the space of all continuous functions on Ω. For $k \in L_\infty(\Omega)$ we define a functional $\mu(k)$ by

$$\mu(k) = \inf_{f \in S(\Omega)} \|k - f\|. \tag{1.2}$$

In [2], Diliberto and Strauss considered the problem of finding a sequence $\{f_n\} \subset S(\Omega)$ so that $\lim_{n\to\infty} \|k - f_n\| = \mu(k)$. They were able to do this and for continuous k their sequence possessed a convergent subsequence. In [1], Aumann showed that their sequence converged if $m = 2$ and k is continuous. The purpose of this note is to extend these results to the case $k \in K(\Omega)$.

The reader should note that $K(\Omega) \neq L_\infty(\Omega)$. To see this let $\Omega_1 = \Omega_2 = [0, 1]$ and let $f(x_1, x_2)$ be given by

$$\begin{aligned} f(x_1, x_2) &= 1 \qquad x_1 \geqslant x_2 \\ &= 0 \qquad x_1 < x_2. \end{aligned} \tag{1.3}$$

Then it is easy to see that

$$\inf_{k \in K(\Omega)} \|f - k\| = \tfrac{1}{2}. \tag{1.4}$$

II

For $k \in L_\infty(\Omega)$ and $1 \leqslant j \leqslant m$ define $H_j(k) \in L_\infty(\Omega_j)$ for μ_j-almost every x_j, by

$$H_j(k)(x_j) = \tfrac{1}{2}\left(\operatorname*{ess\,sup}_{\substack{x_i \in \Omega_i \\ i \neq j}} k(x_1,..., x_m) + \operatorname*{ess\,inf}_{\substack{x_i \in \Omega_i \\ i \neq j}} k(x_1,..., x_m)\right). \tag{2.1}$$

The sequence f_n of Diliberto and Strauss is defined as follows. Let k_n be given by

$$\begin{aligned} k_0 &= k, \\ k_1 &= k_0 - H_1(k_0), \\ k_2 &= k_1 - H_2(k_1), \\ k_{mp+r} &= k_{mp+r-1} - H_r(k_{mp+r-1}), \\ &\text{for} \quad 1 \leqslant r \leqslant m \quad \text{and} \quad p \geqslant 0. \end{aligned} \tag{2.2}$$

We define f_n by

$$f_n = k - k_n. \tag{2.3}$$

The following theorem was proved by Diliberto and Strauss for continuous k. Their proof generalizes directly to the case considered here.

THEOREM 2.1. *For $k \in L_\infty(\Omega)$, let k_n be given by* (2.2). *Then*

$$\lim_{n \to \infty} \|k_n\| = \lim_{n \to \infty} \|k - f_n\| = \mu(k).$$

Moreover, for $n \geqslant 1$, $\|k_n\| \leqslant \|k_{n-1}\|$, and hence,

$$\|f_n\| \leqslant 2\,\|k\|.$$

We list some obvious properties of the functions $H_i(k)$ in the following lemma.

LEMMA 2.1. *Let $k \in L_\infty(\Omega)$ and let i be fixed. Let $\{E_r\}_{r=1}^R$ be a partition of Ω_i into disjoint measurable sets. Let $\phi \in L_\infty(\Omega_i)$ and let $p_r \in L_\infty(\Omega)$ be independent of x_i for $1 \leqslant r \leqslant R$. Then*

(a) $H_i(k + \phi) = H_i(k) + \phi$.

(b) $\|k - H_i(k)\| \leqslant \|k\|$.

(c) $H_i(\phi k) = \phi H_i(k)$.

(d) $H_i(\sum_{r=1}^R X_{E_r} p_r) = \sum_{r=1}^R X_{E_r} H_i(p_r)$.

In (d), *X_{E_r} is the characteristic funtion of the set E_r.*

For $f \in S(\Omega)$, we may write $f = \sum_{i=1}^m \phi_i$ with $\phi_i \in L_\infty(\Omega_i)$. This representation is unique in the sense that if $f = \sum_{i=1}^m \psi_i$, then there are constants δ_i, so that $\sum_{i=1}^m \delta_i = 0$ and $\psi_i + \delta_i = \phi_i$. For $f \in S(\Omega)$ as above and $k \in L_\infty(\Omega)$ we define $Q_k(f) = \sum_{i=1}^m q_i(f)$, where $q_i \in L_\infty(\Omega_i)$ is given by

$$q_i(f) = H_i\left(k - \sum_{j=1}^{i-1} q_j - \sum_{j=i+1}^{m} \phi_j\right). \tag{2.4}$$

Note that each individual q_i depends on the representation $f = \sum_{i=1}^m \phi_i$. However, $Q_k(f)$ does not depend on this representation. Indeed, if $f = \sum_{i=1}^m (\phi_i + \delta_i)$, where the δ_i's are constants such that $\sum_{i=1}^m \delta_i = 0$, let $\hat{q}_i$ be defined by (2.4) with ϕ_i replaced by $\phi_i + \delta_i$. We have $\hat{q}_1 = H_1(k - \sum_{i=2}^m (\phi_i + \delta_2)) = H_1(k - \sum_{i=2}^m \phi_i) - \sum_{i=2}^m \delta_i = q_1 + \delta_1$. Hence $\hat{q}_2 = q_2 - \delta_1 - \sum_{i=3}^m \delta_i = q_2 + \delta_2$. Continuing in this way we obtain $\hat{q}_i = q_i + \delta_i$ for all i, and hence $\sum_{i=1}^m q_i = \sum_{i=1}^{\bar{m}} \hat{q}_i$. Note that Q_k is a continuous map on $S(\Omega)$.

For $k \in L_\infty(\Omega)$, let f_n be defined by (2.3). Fix $p \geqslant 0$ and write $f_{mp} = \sum_{i=1}^m \phi_i$. We have $k_{mp} = k - \sum_{i=1}^m \phi_i$. Hence

$$\begin{aligned} k_{mp+1} &= k_{mp} - H_1(k_{mp}) \\ &= k - \sum_{i=1}^{m} \phi_i - H_1\left(k - \sum_{i=1}^{m} \phi_i\right) \\ &= k - \sum_{i=2}^{m} \phi_i - H_1\left(k - \sum_{i=2}^{m} \phi_i\right) \\ &= k - \sum_{i=2}^{m} \phi_i - q_1(f_{mp}). \end{aligned} \tag{2.5}$$

Continuing we have

$$\begin{aligned} k_{mp+2} &= k_{mp+1} - H_2(k_{mp+1}) \\ &= k - \sum_{i=2}^{m} \phi_i - q_1(f_{mp}) - H_2\left(k - \sum_{i=2}^{m} \phi_i - q_1(f_{mp})\right) \\ &= k - \sum_{i=3}^{m} \phi_i - q_1(f_{mp}) - q_2(f_{mp}). \end{aligned}$$

Finally, we obtain,

$$k_{mp+m} = k - \sum_{i=1}^{m} q_i(f_{mp}) = k - f_{m(p+1)}. \tag{2.7}$$

Hence,

$$f_{mp} = Q_k^p(0); \qquad p \geqslant 1. \tag{2.8}$$

Also by Theorem 2.1, for $k \in L_\infty(\Omega)$ and $f \in S(\Omega)$, we have

$$\|k - Q_k(f)\| \leqslant \|k - f\|. \tag{2.9}$$

Hence, for each k, Q_k maps bounded sets in $S(\Omega)$ into bounded sets in $S(\Omega)$. Moreover, for $p \geqslant 1$

$$\|Q_k^p(0) - k\| \leqslant \|k\|. \tag{2.10}$$

Therefore Q_k maps the set $B_k = \{f \in S(\Omega) \mid \|k - f\| \leqslant 1\}$ into itself and the sequence $\{Q_k^p(0)\}_{p=1}^{\infty}$ is bounded.

THEOREM 2.2. *Let $k \in L_\infty(\Omega)$. Q_k is a compact map on $S(\Omega)$, and therefore is a compact map on B_k. Hence $\{Q_k^p(0)\}_{p=1}^{\infty}$ has a convergent subsequence and the infimum in* (1.2) *is attained.*

Proof. We give the proof for $m = 2$. The proof for arbitrary m is similar. Note first that for $j = 1, 2$ and $k_1, k_2 \in L_\infty(\Omega)$, we have

$$\|H_j(k_1) - H_j(k_2)\| \leqslant \|k_1 - k_2\|. \tag{2.11}$$

If $f(x_1, x_2) = \phi_1(x_1) + \phi_2(x_2)$, then

$$Q_k(f) = H_1(k - \phi_2) + H_2(k - H_1(k - \phi_2)). \tag{2.12}$$

Hence, for any $f \in S(\Omega)$, k_1, $k_2 \in L_\infty(\Omega)$,

$$\|Q_{k_1}(f) - Q_{k_2}(f)\| \leqslant 3\,\|k_1 - k_2\|. \tag{2.13}$$

Let $\varepsilon > 0$. As $k \in K(\Omega)$, we may find finitely many disjoint measurable sets $\{E_r^i\}_{r=1}^R$ in Ω_i such that $\bigcup_{r=1}^R E_r^i = \Omega_i$, and real numbers $\{\alpha_{rs}\}_{r,s=1}^R$, so that

$$\left\| k - \sum_{r,s=1}^{R} \alpha_{rs} \chi_{E_r^1} \chi_{E_s^2} \right\| < \varepsilon/3. \tag{2.14}$$

Now let $\hat{k} = \sum_{r,s=1}^R \alpha_{rs} \chi_{E_r^1} \chi_{E_s^2}$. For $f = \phi_1 + \phi_2 \in S(\Omega)$, we have, by Lemma 2.1, that

$$Q_{\hat{k}}(f) = \sum_{r=1}^{R} \chi_{E_r^1} H_1 \left[\sum_{s=1}^{R} \alpha_{rs} \chi_{E_s^2} - \phi_2 \right] + \sum_{s=1}^{R} \chi_{E_s^2} H_2 \left[\sum_{r=1}^{R} \alpha_{rs} \chi_{E_r^1} - H_1(\hat{k} - \phi_2) \right].$$

As $\chi_{E_s^2}$ and ϕ_2 are independent of x_1, $H_1\,[\sum_{s=1}^R \alpha_{rs} \chi_{E_s^2} - \phi_2]$ is constant for each r. Similarly $H_2[\sum_{r=1}^R \alpha_{rs} \chi_{E_r^1} - H_1(\hat{k} - \phi_2)]$ is constant. Hence $Q_{\hat{k}}$ has finite dimensional range.

We apply (2.13) twice to obtain

$$\|Q_k(f) - Q_{\hat{k}}(f)\| \leqslant 3\,\|k - \hat{k}\| < \varepsilon. \tag{2.15}$$

Hence Q_k is the uniform limit of maps on $S(\Omega)$ which have finite dimensional range. This completes the proof.

We note that Theorem 2.2 is in a sense a converse of a theorem in [3]. Golomb showed, in the case $m = 2$, that if one assumes that the minimum in (1.2) is attained, then Theorem 2.1 holds.

The reader should note that if k is continuous on Ω, so is $Q_k^p(0)$ for each $p \geqslant 1$. Hence if k is continuous the infimum in (1.2) is attained at a continuous $f \in S(\Omega)$.

The remainder of the paper concerns only the case $m=2$. In this case we have

$$Q_k(f)=f+H_1(k-f)+H_2(k-f-H_1(k-f)) \tag{2.16}$$

for $f\in S(\Omega)$. We let U_k denote the set of limit points of the sequence $\{Q_k^p(0)\}_{p=1}^{\infty}$. U_k is non-empty for $k\in K(\Omega)$ by Theorem 2.2. As $H_2(k-Q_k^p(0))=0$ for every $p\geqslant 1$ and $\lim_{p\to\infty}\|k-Q_k^p(0)\|=\lim_{p\to\infty}\|k-Q_k^p(0)-H_1(k-Q_k^p(0))\|=\mu(k)$, we have for $k\in K(\Omega)$ and $f\in U_k$,

$$H_2(k-f)=0 \tag{2.17}$$

and

$$\|k-f\|=\|k-f-H_1(k-f)\|=\mu(k). \tag{2.18}$$

Also note that if $f\in U_k$, $Q_k(f)\in U_k$.

For $m=2$ and $k\in K(\Omega)$ we define

$$\begin{aligned}
P_1(k)&=\operatorname*{ess\,sup}_{x_2\in\Omega_2} k(x_1,x_2),\\
p_1(k)&=\operatorname*{ess\,inf}_{x_2\in\Omega_2} k(x_1,x_2),\\
P_2(k)&=\operatorname*{ess\,sup}_{x_1\in\Omega_1} k(x_1,x_2),\\
p_2(k)&=\operatorname*{ess\,inf}_{x_1\in\Omega_1} k(x_1,x_2).
\end{aligned} \tag{2.19}$$

Hence $H_j(k)=\frac{1}{2}(P_j(k)+p_j(k))$.

We require the following theorem given for the case of continuous k in [1] and [2].

THEOREM 2.3. *Let $m=2$, $k\in K(\Omega)$, and $f\in U_k$, then $H_1(k-f)=0$.*

Proof. If $g\in K(\Omega)$ is such that $H_1(g)=0$, we have, for almost every $x_1\in\Omega$,

$$-P_1(H_2(g))\leqslant H_1(g-H_2(g))\leqslant -p_1(H_2(g)). \tag{2.20}$$

Similarly if $k\in K(\Omega)$ is such that $H_2(h)=0$ we have, for almost every $x_1\in\Omega_1$,

$$-P_2(H_1(h))\leqslant H_2(h-H_1(h))\leqslant -p_2(H_1(h)). \tag{2.21}$$

If $\{k_n\}$ is the sequence given by (2.2), inequalities (2.20) and (2.21) imply that, for $n \geqslant 1$,

$$-P_2(H_1(k_{2n})) \leqslant p_1(H_2(k_{2n+1})) \leqslant -P_2(H_1(k_{2n+1})), \tag{2.22}$$

$$P_1(H_2(k_{2n+1})) \leqslant -p_2(H_1(k_{2n})) \leqslant P_1(H_2(k_{2n-1})). \tag{2.23}$$

Hence there are real numbers α and β so that for every $f \in U_k$ we have

$$\begin{aligned} \alpha &= P_2(H_1(k-f)) = -p_1(H_2(k-f-H_1(k-f))), \\ \beta &= P_1(H_2(k-f-H_1(k-f))) = -p_2(H_1(k-f)). \end{aligned} \tag{2.24}$$

The proof will be complete if we show that $\alpha = \beta = 0$. To do this we extend the ideas of [1]. Without loss of generality we may assume that $\alpha \leqslant \beta$.

Let $\varepsilon > 0$, for $f \in U_k$ we define a measurable set $E(f) \leqslant \Omega_2$ by

$$E(f) = \{x_2 \in \Omega_2 \mid H_2(k-f-H_1(k-f))(x_2) \geqslant \beta - \varepsilon\}. \tag{2.25}$$

We define a functional τ on U_k by

$$\tau(f) = \operatorname*{ess\,sup}_{x_2 \in E(f)} P_2(k-f-H_1(k-f))(x_2). \tag{2.26}$$

We show that for every $g \in U_k$,

$$\tau(Qg) \geqslant \tau(g) + 2\beta - 2\varepsilon. \tag{2.27}$$

For $g \in U_k, f = Qg$, and $x_2 \in E(f)$ we have, as in [1]

$$P_2(k-f-H_1(k-f)) \geqslant P_2(k-f) + \beta - \varepsilon. \tag{2.28}$$

Hence, if $S = \operatorname{ess\,sup}_{x_2 \in E(f)} P_2(k-f)$, we must have

$$S + \beta \geqslant \tau(f) \geqslant S + \beta - \varepsilon. \tag{2.29}$$

Moreover there is $F(f) \subseteq \Omega_1$ with $\mu_1(F(f)) > 0$ such that for $(x_1, x_2) \in F(f) \times E(f)$ we have

$$(k-f-H_1(k-f))(x_1, x_2) \geqslant S + \beta - \varepsilon. \tag{2.30}$$

Hence, for $x_1 \in F(f)$, we must have

$$H_1(k-f) \leqslant -\beta + \varepsilon. \tag{2.31}$$

As $\mu_2(E(f))>0$ and $H_1(k-f-H_1(k-f))=0$ there is $G(f)\subseteq \Omega_2$ with $\mu_2(G(f))>0$ so that for $(x_1,x_2)\in F(f)\times G(f)$,

$$(k-f-H_1(k-f))(x_1,x_2)\leqslant -S-\beta+\varepsilon. \tag{2.32}$$

Therefore by (2.31) we have, for $(x_1,x_2)\in F(f)\times G(f)$,

$$\begin{aligned}(k-f)(x_1,x_2)&\leqslant -S-2\beta+2\varepsilon\\ &\leqslant -\tau(f)-\beta+2\varepsilon.\end{aligned} \tag{2.33}$$

Let $\sigma(f)=-\text{ess inf}_{x_1\in F(f)}\,(k-f)(x_1,x_2)$. We have

$$-\sigma(f)\leqslant -\tau(f)-\beta+2\varepsilon. \tag{2.34}$$

A similar argument will show that

$$\sigma(f)=\sigma(Qg)\geqslant \tau(g)+\beta-2\varepsilon. \tag{2.35}$$

Hence,

$$\tau(Q_k\, g)\geqslant \tau(g)+2\beta-4\varepsilon. \tag{2.36}$$

Therefore $\beta\leqslant 0$, as if $\beta>0$ and $0<\varepsilon<\beta/2$, we must have, for all $p\geqslant 1$, $g\in U_k$,

$$\mu(k)\geqslant \tau(Q_k^p\, g)\geqslant p\beta. \tag{2.37}$$

By (2.16) and (2.17) we have, as $\alpha+\beta\leqslant 0$,

$$\begin{aligned}\mu(k)&=P_1P_2(k-Qf)\\ &=P_1P_2(k-f-H_1(k-f)-H_2(k-f-H_1(k-f))).\\ &\geqslant P_1P_2(k-f)-(\alpha+\beta)\\ &=\mu(k)-(\alpha+\beta).\end{aligned} \tag{2.38}$$

Hence $\alpha+\beta=0$ and $\alpha=\beta=0$. This completes the proof.

THEOREM 2.4. *Let $m=2$, $k\in K(\Omega)$. The sequence $\{k_n\}_{n=1}^{\infty}$ given by* (2.2) *converges in $L_\infty(\Omega)$.*

Proof. Let k_* be any limit point of the sequence $\{k_{2n}\}_{n=1}^{\infty}$. We write $k=k_*+\phi_1+\phi_2$ with $\phi_i\in L_\infty(\Omega_i)$. Then for $n\geqslant 2$ we have $k_{2n}=k_*+\phi_1^{(n)}+\phi_2^{(n)}$,

$$\begin{aligned}\phi_1^{(n)}&=-H_1(k_*+\phi_2^{(n-1)}),\\ \phi_2^{(n)}&=-H_2(k_*+\phi_1^{(n)}).\end{aligned} \tag{2.39}$$

By Theorem 2.3, $H_i(k_*)=0$, for $i=1\ 2$, hence,

$$\begin{aligned} -p_2(\phi_2^{(n)}) &\leqslant P_1(\phi_1^{(n)}) \leqslant -p_2(\phi_2^{(n-1)}), \\ P_2(\phi_2^{(n)}) &\leqslant -p_1(\phi_1^{(n)}) \leqslant P_2(\phi_2^{(n-)}). \end{aligned} \tag{2.40}$$

Now by Theorem 2.2, there is a subsequence $\{k_{2nj}\}_{j=1}^{\infty}$ of $\{k_{2n}\}_{n=1}^{\infty}$ which converges to k_* in $L_\infty(\Omega)$. This means that if $\bar{\phi}_i^{(j)}=\phi_i^{(nj)}$ we have $\lim_{j\to\infty} \|\bar{\phi}_1^{(j)}+\bar{\phi}_2^{(j)}\|=0$. This, together with (2.40) imply that there is a real number c so that

$$\lim_{j\to\infty} \bar{\phi}_1^{(j)}=c=-\lim_{j\to 8} \bar{\phi}_2^{(j)}. \tag{2.41}$$

Now let $\varepsilon>0$. Choose j so that $j\leqslant j_0$ implies that

$$\|\bar{\phi}_1^{(j)}-c\|<\frac{\varepsilon}{2}, \qquad \|\bar{\phi}_2^{(j)}+c\|<\frac{\varepsilon}{2}. \tag{2.42}$$

Inequality (2.4) implies that for all $n\geqslant n_{j_0}$, and almost every $x_1\in\Omega_1$, and $x_2\in\Omega_2$,

$$\begin{aligned} c-\frac{\varepsilon}{2} &\leqslant \phi_1^{(n)} \leqslant c\frac{\varepsilon}{2}, \\ -c-\frac{\varepsilon}{2} &\leqslant \phi_2^{(n)} \leqslant -c+\frac{\varepsilon}{2}. \end{aligned} \tag{2.43}$$

Hence $\|\phi_1^{(n)}+\phi_2^{(n)}\|\leqslant\varepsilon$ for $n\geqslant n_{j_0}$, and therefore

$$\lim_{n\to\infty} k_{2n}=k_*. \tag{2.44}$$

As $\lim_{n\to\infty}\|k_{2n}-k_{2n+1}\|=\lim_{n\to\infty}\|H_1(k_{2n})\|=\|H_1(k_*)\|=0$, $\lim_{n\to\infty} k_{2n+1}=k_*$. This completes the proof.

Appendix

Here we prove that the subspace $S(\Omega)$ given by Eq. (1.1) is closed.

We define continuous projections $\{P_i\}_{i=0}^{m}$, $\{Q_i\}_{i=1}^{m}$, and $\{R_i\}_{i=1}^{m}$ as follows

$$P_0 f=\frac{1}{\mu(\Omega)}\int_\Omega f(x)\,dx, \tag{A1}$$

$$Q_i f=\frac{1}{\mu_i(\Omega_i)}\int_{\Omega_i} f(x)\,dx_i, \tag{A2}$$

$$R_i f = \left(\prod_{j \neq j} Q_j\right) f, \tag{A3}$$

$$P_i f = (R_i - P_0) f, \qquad 1 \leqslant i \leqslant m, \tag{A4}$$

All of the above projections commute. Note that $R_i f$ and $P_i f$ depend only on x_i. $Q_i f$ is independent of x_i. We have

$$R_i Q_i = Q_i R_i = P_0, \qquad 1 \leqslant i \leqslant m, \tag{A5}$$

$$P_0 Q_i = Q_i P_0 = R_i P_0 = P_0 R_i = P_0. \tag{A6}$$

For $1 \leqslant i \leqslant m$, by Eq. (A6).

$$P_i P_0 = (R_i - P_0) P_0 = R_i P_0 - P_0 = 0. \tag{A7}$$

For $1 \leqslant i, j \leqslant m$, we have

$$R_i R_j = \begin{Bmatrix} P_0 & i \neq j \\ R_j & i = j \end{Bmatrix} = R_j R_i \tag{A8}$$

and hence,

$$\begin{aligned} P_i P_j &= (R_i - P_0)(R_j - P_0) \\ &= R_i R_j - P_0 R_j - R_i P_0 + P_0 \\ &= R_i R_j - P_0 = \delta_{ij} P_j. \end{aligned} \tag{A9}$$

Therefore, the operator P given by

$$P = \sum_{j=0}^{m} P_j \tag{A10}$$

is a continuous (and hence closed) projection on $L^{q}(\Omega)$ and [4, p. 241] hence has closed range. We show that this range is $S(\Omega)$.

As $P_0 f$ is constant for all $f \in L^{q}(\Omega)$ and $P_j f$ depends only on x_j, $\text{Ran}(P) \leqslant S(\Omega)$ by definition. We will be done if we show that $P\phi = \phi$ for all $\phi \in S(\Omega)$.

By the definition of S, it will suffice to show that if $\phi_k \in L^{q}(\Omega)$ depends only upon x_k, then $P\phi_k = \phi_k$. We have

$$Q_k \phi_k = P_0 \phi_k = R_i \phi_k \qquad \text{if} \quad i \neq k, \tag{A11}$$

$$Q_i \phi_k = R_k \phi_k = \phi_k \qquad \text{if} \quad i \neq k. \tag{A12}$$

Hence,

$$P_i \phi_k = 0, \qquad i \neq k, \quad i \geqslant 1, \tag{A13}$$

$$P_k \phi_k = \phi_k - P_0(\phi_k). \tag{A14}$$

Therefore,

$$P\phi_k = \sum_{j=0}^{m} P_j \phi_k = P_0 \phi_k + P_k \phi_k = \phi_k \tag{A15}$$

and the proof is complete.

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